SPECIMEN MATHEMATICUM,

DE

Methodo Soliditates Corporum duplici integratione eruendi.

Venia Ampl. Fac. Philos. Aboëns.

Ad publicum Examen deferunt

AUCTOR

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ET

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In Auditorio Majori die 3 Decembr. 1800.

h. a. m. s.

ABOÆ, in Officina FRENCKELLIANA.

I.

KONUNGENS

TRO-MAN,

ÖFVERSTEN FÖR TAVASTEHUS LÅNS LINIE-REGIMENTE, OCH JÄGARE-CORPS, SAMT RIDDAREN AF DEN KONGL. SVÄRDS-ORDEN,

HÖGVÄLBORNE HERREN,

HERR

HANS HENR. CRISPENBERG,

Tilägnas dessa blad i djupaste ödmjukhet

af

HÖGVÄLBORNE HERR ÖFVERSTENS OCH *RIDDARENS*

allerödmjukaste tjenare

JOHAN FRED. AHLSTEDT.

§. I.

Theoria Solidorum, etsi dudum jam, disquisitionibus quibus abundat elegantissimis, cultores sibi allicuit sollertissimos, detectæ tamen olim a sagacissimis Geometris Methodo Fluxionum, pariter ac universa Mathesis incrementa sua debet splendidissima. Etenim Corporum per motum perspicienda Genesis, iis indagandis adeo videtur esse necessaria, ut sublata hac notione, vix pateat modus, quo recondita illorum innotescat natura. Genuina vero hæc Methodus, data solummodo relatione inter coordinatas solidi cujusvis exprimentes affectiones, viam sternit amplissimam ad cognitionem ejus, exiguo quidem labore, perveniendi. Juvat ergo repræsentare Corpus quodcunque *ABQGP* (*Fig. I.*) motu superficiei planæ *BQGP* esse genitum, quæ, secundum lineam *AC*, sibi semper parallele moveri pergens, eodem tempusculi momento, quo axis *AP* acquirit augmentum *Pp*, fluxionem puncti *P* repræsentans, hujus generabit fluxionem prismaticam *BQGhPp*. Cujus valor si cognitus spectatur, methodo egregie exculta ipsas soliditates investigare solent harum rerum cultores. Sin autem ipsum hoc elementum non constet, pari omnino ratione idem considerare licebit motu plani *QMNr* esse generatum; adeoque acquirente linea fluente *PM* augmentum elementare *Mm*, ipsum planum percurret spatiolum *Qn*, fluxionem elementi *BN* definiens & parallelipipedi habens formam. Hinc alia oritur Methodus corporum eruendi soliditates, quæ, cum & latius patet quam supra memorata, quoniam non solum *tota* hoc artificio innotescit corporis *ABPG fluxio*, verum etiam quæcunque ejus *portio* *BN* a nostro dependens lubitu, & aream sectionis *BQGP* notam non supponit, illi merito erit anteponenda.

Ex dato autem valore columellæ *Qn*, soliditatis investigandæ rationem exposuit *Cel. Dom.* LEON. EULERUS *in Diss. de Formulis integralibus duplicatis, Nov. Comment. Acad. Petrop. Tom. XIV. P. I. inserta;* quam nondum ad summum evectam fastigium, specimine qualicunque illustrare, nostraque conamina moderatæ L. C. censuræ submittere constituimus.

§. II.

PROBLEMA. *Datis angulis* $APG = m$, BPG ($= QMG$) $= p$ *&* APB ($= QMS$) $= q$, *quibus inclinantur ternæ sectiones* APG, BGP, ABP, *valorem Columellæ*

mellæ Qn, parallelogrammulo MNnm insistentis, invenire.

Sint, $AP = x$, $PM = y$, $QM = z$, $Pp = MN = mn = dx$, $Mm = Nn = dy$. Demissis e puncto Q normalibus QR in planum APG, QT in PG & QS in NM productam axi AC parallelam, jungantur RM, RT, RS, quæ ultima producta occurrat ipsi PG in puncto U, & a puncto M agatur normalis Mo in mn.

His præmissis, erit in triangulo Mmo $1 (= Rad) : Sin\ m :: dy : Mo = dy\ Sin\ m$, qua in $mn = dx$ ducta obtinetur areola $MNnm = dxdy\ Sin\ m$.

Præterea, quoniam anguli QRM, QRS, QRT, QSM, QTM sunt recti *(per constr.)*, erit $QM^2 (= QS^2 + MS^2) = QR^2 + RS^2 + MS^2 (= QT^2 + MT^2) = QR^2 + RT^2 + MT^2 = QR^2 + RM^2$, sive $MR^2 = RS^2 + MS^2 = RT^2 + MT^2$; adeoque anguli $RSM = RTM = 90°$. Hinc in triangulo MSU erit, $1 : tang\ m \left(= \frac{Sin\ m}{Cos\ m}\right) :: MS : SU = \frac{MS \times Sin\ m}{Cos\ m}$, & ob $MSU \sim RTU$, $SU : MU :: TU : RU = \frac{MU \times TU}{SU}$. Est vero $1 : Cos\ q :: z : MS = z\ Cos\ q$; $1 : Cos\ p :: z : MT = z\ Cos\ p$ & $1 : Sin\ q :: z : QS = z\ Sin\ q$, erit ergo $MU =$

$$\sqrt{MS^2 + SU^2} = \frac{z\ Cos\ q \sqrt{Sin\ m^2 + Cos\ m^2}}{Cos\ m} = \frac{z\ Cos\ q}{Cos\ m}\ \&$$

$$TU = MU - MT = \frac{z\ (Cos\ q - Cos\ m\ Cos\ p)}{Cos\ m},$$ quare erit

 RU

$RU = \frac{z(Cos\,q - Cos\,m\,Cos\,p)}{Sin\,m\,Cos\,m}$. Hinc porro obtinetur $RS = SU - RU = \frac{z(Cos\,p - Cos\,m\,Cos\,q)}{Sin\,m}$, unde demum resultat $QR = \sqrt{QS^2 - RS^2} =$

$$\frac{z\sqrt{Sin\,m^2\,Sin\,q^2 - Cos\,p^2 + 2\,Cos\,m\,Cos\,p\,Cos\,q - Cos\,m^2\,Cos\,q^2}}{Sin\,m}$$

$$= z\,\frac{\sqrt{Sin\,m^2 - Cos\,p^2 + 2\,Cos\,m\,Cos\,p\,Cos\,q - Cos\,q^2}}{Sin\,m},$$ ob $Sin\,m^2\,Sin\,q^2 - Cos\,m^2\,Cos\,q^2 = Sin\,m^2 - Cos\,q^2$. Ex elementis autem constat esse soliditatem parallelipipedi Qn æqualem producto ex area basis $MNnm$ in altitudinem QR, quæ cum notæ sunt, erit $Qn = zdxdy\sqrt{Sin\,m^2 - Cos\,p^2 + 2\,Cos\,m\,Cos\,p\,Cos\,q - Cos\,q^2}$.

Coroll. Quod si tria hæc plana se mutuo ad angulos rectos decussent, erit $Qn = zdxdy$, ob $Cos\,m = Cos\,p = Cos\,q = 0$ & $Sin\,m = 1$ seu sinui toti.

§. III.

Formulæ hujusmodi differentialis:

$$zdxdy\sqrt{Sin\,m^2 - Cos\,p^2 + 2\,Cos\,m\,Cos\,p\,Cos\,q - Cos\,q^2}$$

valor finitus *duplici* investigari solet *integratione*, unde $\iint zdxdy\sqrt{Sin\,m^2 - Cos\,p^2 + 2\,Cos\,m\,Cos\,q\,Cos\,p - Cos\,q^2}$, ipsum solidum repræsentanti *duplex* præfigitur integrationis *signum*. Cum autem æquatio pro solido data supponitur, variatio ipsius z per variabilis x & y defi-

definietur, quæ functione quacunque harum expressa in formula est adhibenda. Binarum vero actu instituendarum integrationum, *alteri* sola y est obnoxia, spectando x tanquam invariatam; qua peracta ipsi y tribuatur *valor ultimus*, quem per totam x attingere valebit, qui ergo aut constantem exprimet quantitatem aut functionem quamcunque ipsius x involvet, unde in *altera integratione* sola x restabit variabilis.

Si vero ordine inverso primo quantitas y constans habeatur & integrale . . . . . .

$$\int z\,dx\,\sqrt{Sin\,m^2 - Cos\,p^2 + 2\,Cos\,m\,Cos\,p\,Cos\,q - Cos\,q^2}$$

per terminos præscriptos extendatur, id deinceps ut functio ipsius y spectari & solidum quæsitum . .

$$\int dy \int z\,dx\,\sqrt{Sin\,m^2 - Cos\,p^2 + 2\,Cos\,m\,Cos\,p\,Cos\,q - Cos\,q^2}$$

inveniri poterit. Utraque demum integratione quantitas introducitur *arbitraria*, quæ functionem quamcunque ipsarum x & y invehere potest.

Quum ergo æquatio pro Solido innotescit, qua coordinatæ z relatio per x & y exprimitur, totum conficitur negotium in integranda formula traditis hisce regulis invenienda; quare variis exemplis hanc rem illustrare juvabit. Corpora autem, quorum sectiones sunt *lineæ secundi ordinis* sive *sectiones conicæ*, præ alia enodare placet, quorum tum soliditates *integras*, tum *portionem* basi rectangulari insistentem exhibebimus. Cumque solida secundi ordinis æquatione definita ad *sex genera* reducere liceat, (*conf.* Eu-

 LERI

LERI *Introd. in Anal. Infinit.*) ſequentibus exemplis totum hunc ordinem complectemur.

Ex. 1. Sumta origine Abſciſſarum in Centro *Elliptoidis*, relationem trium coordinatarum *orthogonalium* iſta exhibebit æquatio: $A^2x^2 + B^2y^2 + C^2z^2 = \alpha^2$. Valor ergo ipſius z definietur æquatione $z = \frac{\sqrt{\alpha^2 - A^2x^2 - B^2y^2}}{C}$, unde formula $zdxdy$, pro *coordinatis orthogonalibus* (*Coroll.* §. 2.), hanc induet formam: $\frac{dxdy\sqrt{\alpha^2 - A^2x^2 - B^2y^2}}{C}$, qua ita integrata, ut ſola y variabilis ſpectetur, prodibit $\frac{dx}{C}\int dy\sqrt{\alpha^2 - A^2x^2 - B^2y^2} = \frac{dx}{2C}\int\frac{(\alpha^2 - A^2x^2 - 2B^2y^2)dy}{\sqrt{\alpha^2 - A^2x^2 - B^2y^2}} + \frac{dx}{2C}\int\frac{(\alpha^2 - A^2x^2)dy}{\sqrt{\alpha^2 - A^2x^2 - B^2y^2}} = \frac{ydx\sqrt{\alpha^2 - A^2x^2 - B^2y^2}}{2C} + \frac{(\alpha^2 - A^2x^2)dx}{2BC} ArcSin\frac{By}{\sqrt{\alpha^2 - A^2x^2}}$ integrali ita correcto, ut poſito $y = 0$ ipſum quoque evaneſcat. Antequam nova jam ſuſcipiatur integratio, extendatur y ad terminum ultimum, quem per totam x attingere valebit. Si ergo integra *Elliptoidis* ſoliditas quæratur, porrigetur y usque ad *Ellipſin* $AE'G$ quo pacto erit $y = \frac{\sqrt{\alpha^2 - A^2x^2}}{B}$. Tributo hoc valore pro y, erit $\int\frac{ydx\sqrt{\alpha^2 - A^2x^2 - B^2y^2}}{2C}$ +

+

$$+\int\frac{(\alpha^2-A^2x^2)dx}{2BC}\,Arc\,Sin\frac{By}{\sqrt{\alpha^2-A^2x^2}}=\int\frac{(\alpha^2-A^2x^2)\pi dx}{4BC}$$

(quia membrum primum evanescit, & angulus posterius afficiens abit in angulum rectum, cujus mensura est quadrans peripheriæ circularis $=\frac{\pi}{2}$) $=\frac{(3\alpha^2-A^2x^2)\pi x}{12BC}$ $+C$. Si vero solidum a centro inchoët constans addenda evanescit. Extenso x ad A, quo casu erit $x=\frac{\alpha}{A}$, prodibit *octans Elliptoidis* $=\frac{\alpha^3\pi}{6ABC}$.

Quod si *portionis* tantum, quæ rectangulo $CDEF$ insistit, *soliditatem* investigare velimus, ipsi y in altera integratione constans tribuetur valor, scil. $y=CD=f$, unde formulæ $\int\frac{(\alpha^2-A^2x^2)dx}{2BC}\,Arc\,Sin\frac{Bf}{\sqrt{\alpha^2-A^2x^2}}+\int\frac{fdx\sqrt{\alpha^2-A^2x^2-B^2f^2}}{2C}$ dabunt solidum quæsitum ita correctum, ut posito $x=0$ ipsum quoque in nihilum redigatur. Evolventur ergo binæ hæ formulæ.

Cum sit: $d.\frac{(3\alpha^2-A^2x^2)x}{6BC}\,Arc\,Sin\frac{Bf}{\sqrt{\alpha^2-A^2x^2}}=\frac{(\alpha^2-A^2x^2)dx}{2Bf}\,Arc\,Sin\frac{Bf}{\sqrt{\alpha^2-A^2x^2}}+\frac{A^2fx^2(3\alpha^2-A^2x^2)dx}{6C(\alpha^2-A^2x^2)\sqrt{\alpha^2-A^2x^2-B^2f^2}}$, erit integrando

$\int(\alpha^2$

$$\int\frac{(\alpha^2 - A^2x^2)\,dx}{2BC}\ \text{Arc Sin}\ \frac{Bf}{\sqrt{\alpha^2 - A^2x^2}} =$$

$$\frac{(3\alpha^2 - A^2x^2)\,x}{6BC}\ \text{Arc Sin}\ \frac{Bf}{\sqrt{\alpha^2 - A^2x^2}} -$$

$\int\frac{A^2fx^2(3\alpha^2 - A^2x^2)\,dx}{6C(\alpha^2 - A^2x^2)\sqrt{\alpha^2 - A^2x^2 - B^2f^2}}$, cujus integralis postrema pars dividendo numeratorem & denominatorem per $\alpha^2 - A^2.x^2$, transformabitur in:

$$\int\frac{(2\alpha^2 - A^2x^2)\,f\,dx}{6C\sqrt{\alpha^2 - A^2x^2 - B^2f^2}} - \int\frac{\alpha^4 f\,dx}{3C(\alpha^2 - A^2x^2)\sqrt{\alpha^2 - A^2x^2 - B^2f^2}}$$

quibus si addatur membrum $\int\frac{f\,dx\sqrt{\alpha^2 - A^2x^2 - B^2f^2}}{2C}$

$\left(= \int\frac{3(\alpha^2 - A^2x^2 - B^2f^2)\,f\,dx}{6C\sqrt{\alpha^2 - A^2x^2 - B^2f^2}}\right)$ erit:

$$\int\frac{(\alpha^2 - A^2x^2)\,dx}{2BC}\ \text{Arc Sin}\ \frac{Bf}{\sqrt{\alpha^2 - A^2x^2}}$$

$$+ \int\frac{f\,dx\sqrt{\alpha^2 - A^2x^2 - B^2f^2}}{2C} =$$

$$\frac{(3\alpha^2 - A^2x^2)\,x}{6BC}\ \text{Arc Sin}\ \frac{Bf}{\sqrt{\alpha^2 - A^2x^2}} +$$

$$\int\frac{(\alpha^2 - 2A^2x^2 - B^2f^2)\,f\,dx}{3C\sqrt{\alpha^2 - A^2x^2 - B^2f^2}} + \int\frac{(3\alpha^2 - B^2f^2)\,f\,dx}{6C\sqrt{\alpha^2 - A^2x^2 - B^2f^2}}$$

$- \int\frac{\alpha^4 f\,dx}{3C(\alpha^2 - A^2x^2)\sqrt{\alpha^2 - A^2x^2 - B^2f^2}}$, membris ad integrategrationem rite dispositis. Est autem

$$\int\frac{(\alpha^2-2A^2x^2-B^2f^2)f\,dx}{3C\sqrt{\alpha^2-A^2x^2-B^2f^2}}=\frac{fx\sqrt{\alpha^2-A^2x^2-B^2f^2}}{3C}$$

atque $\int\frac{(3\alpha^2-B^2f^2)f\,dx}{6C\sqrt{\alpha^2-A^2x^2-B^2f^2}}=$ $\frac{(3\alpha^2-B^2f^2)f}{6AC}\,\text{Arc Sin}\,\frac{Ax}{\sqrt{\alpha^2-B^2f^2}}$. At quo ultimi membri integrale innotescat, juvat posuisse $z=\frac{ABfx}{\sqrt{(\alpha^2-A^2x^2)(\alpha^2-B^2f^2)}}$, unde eliciuntur $dx=\frac{\alpha B^2f^2\sqrt{\alpha^2-B^2f^2}.\,dz}{A(B^2f^2+(\alpha^2-B^2f^2)z^2)^{\frac{3}{2}}}$, $\alpha^2-A^2x^2=\frac{\alpha^2B^2f^2}{B^2f^2+(\alpha^2-B^2f^2)z^2}$ & $\sqrt{\alpha^2-A^2x^2-B^2f^2}=\frac{Bf\sqrt{\alpha^2-B^2f^2}.\sqrt{1-z^2}}{\sqrt{B^2f^2+(\alpha^2-B^2f^2)z^2}}$ quibus substitutis valoribus obtinebitur $-$

$$\int\frac{\alpha^4f\,dx}{3C(\alpha^2-A^2x^2)\sqrt{\alpha^2-A^2x^2-B^2f^2}}=-\frac{\alpha^3}{3ABC}\int\frac{dz}{\sqrt{1-z^2}}$$

$=-\frac{\alpha^3}{3ABC}\,\text{Arc Sin}\,z$ (seu restituto valore pro z)

$$=-\frac{\alpha^3}{3ABC}\,\text{Arc Sin}\,\frac{ABfx}{\sqrt{(\alpha^2-A^2x^2)(\alpha^2-B^2f^2)}}.$$

Quare facto $x=e=CF$, erit soliditas:

$$\frac{(3\alpha^2-A^2e^2)e}{6BC}\,\text{Arc Sin}\,\frac{Bf}{\sqrt{\alpha^2-A^2e^2}}+\frac{ef\sqrt{\alpha^2-A^2e^2-B^2f^2}}{3C}$$

$+ \frac{(3\alpha^2 - B^2 f^2) f}{6AC}$ Arc Sin $\frac{Ae}{\sqrt{\alpha^2 - B^2 f^2}}$ —
$\frac{\alpha^3}{3ABC}$ Arc Sin $\frac{ABef}{\sqrt{(\alpha^2 - A^2 e^2)(\alpha^2 - B^2 f^2)}}$.

Si rectanguli terminus E porrigatur usque ad E', erit $e = \frac{\sqrt{\alpha^2 - B^2 f^2}}{A}$, quo valore pro e adhibito, membrum algebraicam evanescit & anguli reliquas afficientes evadunt recti. Hinc ergo soliditas portionis rectangulo $CDE'F'$ insistentis $=$

$$\left(\frac{(2\alpha^2 + B^2 f^2)\sqrt{\alpha^2 - B^2 f^2} + (3\alpha^2 - B^2 f^2) Bf - 2\alpha^3}{ABC}\right)\frac{\pi}{12}.$$

Ex. 2. Superficierum *Elliptico-Hyperbolicarum* naturam hæc exprimit æquatio: $B^2 y^2 - A^2 x^2 + C^2 z^2 = \alpha^2$. Hinc cum sit $z dx dy = \frac{dx dy \sqrt{\alpha^2 + A^2 x^2 - B^2 y^2}}{C}$, erit integrando per y, $\frac{dx}{C}\int dy \sqrt{\alpha^2 + A^2 x^2 - B^2 y^2} = \frac{y dx \sqrt{\alpha^2 + A^2 x^2 - B^2 y^2}}{2C}$ $+ \frac{(\alpha^2 + A^2 x^2) dx}{2BC}$ Arc Sin $\frac{By}{\sqrt{\alpha^2 + A^2 x^2}}$. Exprimet ergo hoc integrale, portionem BN fluxionis solidi $ABQPG$, rectangulo PN insistentem, cujus terminus si extendatur ad quævis puncta Hyperbolæ, erit $y = \frac{\sqrt{\alpha^2 + A^2 x^2}}{B}$, qua functione pro y inserta, integratio repe-

repetita dabit $\int \frac{(\alpha^2 + A^2 x^2)\pi dx}{4BC} = \frac{(3\alpha^2 + A^2 x^2)\pi x}{12BC}$.
Facto $x = e$, prodibit *quadrans ſolidi Elliptico-hyperbolici* $= \frac{(3\alpha^2 + A^2 e^2) e\pi}{12BC}$.

Quo autem portionis rectangulo $CDEF$ inſiſtentis ſoliditas innoteſcat, fiat $y = f$, dum altera ſuſcipiatur integratio, unde formulæ evadunt . . .

$$\int \frac{(\alpha^2 + A^2 x^2)dx}{2BC} ArcSin \frac{Bf}{\sqrt{\alpha^2 + A^2 x^2}} + \int \frac{fdx \sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}}{2C}$$

$$= \frac{(3\alpha^2 + A^2 x^2)x}{6BC} ArcSin \frac{Bf}{\sqrt{\alpha^2 + A^2 x^2}} + \int \frac{(\alpha^2 + 2A^2 x^2 - B^2 f^2) fdx}{3C\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}}$$

$$+ \int \frac{(3\alpha^2 - B^2 f^2) fdx}{6C\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}} - \int \frac{\alpha^4 fdx}{3C(\alpha^2 + A^2 x^2)\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}},$$

ſi quidem pari modo ac in exemplo ſuperiore membra diſponantur. Sed eſt $\int \frac{(\alpha^2 + 2A^2 x^2 - B^2 f^2) fdx}{3C\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}} = \frac{fx\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}}{3C}$, quare duo tantum ultima membra ſuperſunt integranda. Prioris integrale quo habeatur, fiat $Ax + \sqrt{\alpha^2 + A^2 x^2 - B^2 f^2} = z$, unde obtinentur, $dx = \frac{(\alpha^2 - B^2 f^2 + z^2)dz}{2Az^2}$ & $\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2} = \frac{\alpha^2 - B^2 f^2 + z^2}{2z}$, hinc erit $\int \frac{(3\alpha^2 - B^2 f^2) fdx}{6C\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}} =$

 $(3\alpha^2$

$\frac{(3\alpha^2 - B^2 f^2)f}{6AC} \int \frac{dz}{z} = \frac{(3\alpha^2 - B^2 f^2)f}{6AC} \, Log \, (Ax + \sqrt{\alpha^2 + A^2 x^2 - B^2 f^2})$ restituto valore pro $Log \, z$ sive $\int \frac{dz}{z}$. Membrum autem ultimum rationale redditur substituendo $\frac{ABfx + \alpha\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}}{ABfx - \alpha\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}} = v$, unde inveniuntur: $dx = \frac{2\alpha B^2 f^2 \sqrt{B^2 f^2 - \alpha^2}\,(v-1)\,dv}{A(\alpha^2 (v+1)^2 - B^2 f^2 (v-1)^2)^{\frac{3}{2}}}$, $\alpha^2 + A^2 x^2 = \frac{4\alpha^2 B^2 f^2 v}{\alpha^2 (v+1)^2 - B^2 f^2 (v-1)^2}$ & $\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2} = \frac{Bf\sqrt{B^2 f^2 - \alpha^2}\,(v-1)}{\sqrt{\alpha^2 (1+v)^2 - B^2 f^2 (v-1)^2}}$, quibus insertis valoribus, evadit $-\int \frac{\alpha^4 f dx}{3C(\alpha^2 + A^2 x^2)\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}} = -\frac{\alpha^3}{6ABC}\int \frac{dv}{v} = -\frac{\alpha^3}{6ABC} Log\left(\frac{ABfx + \alpha\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}}{ABfx - \alpha\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}}\right)$

Quare collectis integralibus inventis erit $\int \frac{(\alpha^2 + A^2 x^2)dx}{2BC} \, ArcSin \frac{Bf}{\sqrt{\alpha^2 + A^2 x^2}} + \int \frac{fdx\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}}{2C}$ $= \frac{(3\alpha^2 + A^2 x^2)x}{6BC} \, ArcSin \frac{Bf}{\sqrt{\alpha^2 + A^2 x^2}} + \frac{fx\sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}}{3C} + \frac{(3\alpha^2 - B^2 f^2)x}{6BC} \, Log\,(Ax + \sqrt{\alpha^2 + A^2 x^2 - B^2 f^2}) -$

$\frac{\alpha^3}{6ABC} Log \left(\frac{ABfx + \alpha\sqrt{\alpha^2 + A^2x^2 - B^2f^2}}{ABfx - \alpha\sqrt{\alpha^2 + A^2x^2 - B^2f^2}}\right) + C$. Si rectanguli terminus E curvam attingat, integrale constituto $x = \frac{\sqrt{B^2f^2 - \alpha^2}}{A}$ nihilo æquabitur, unde $C = -$
$\frac{(2\alpha^2 + B^2f^2)\sqrt{B^2f^2 - \alpha^2}.\pi}{12ABC} - \frac{(3\alpha^2 - B^2f^2)f}{12AC} Log(B^2f^2 - \alpha^2)$;
quare facto $x = e$ erit solidum rectangulum $CDE'F'$ obtegens $= \frac{(3\alpha^2 + A^2e^2)e}{6BC} Arc\,Sin \frac{Bf}{\sqrt{\alpha^2 + A^2e^2}} +$
$\frac{ef\sqrt{\alpha^2 + A^2e^2 - B^2f^2}}{3C} + \frac{(3\alpha^2 - B^2f^2)f}{6AC} \times$
$Log \left(\frac{Ae + \sqrt{\alpha^2 + A^2e^2 - B^2f^2}}{\sqrt{B^2f^2 - \alpha^2}}\right) - \frac{\alpha^3}{6ABC} \times$
$Log\left(\frac{ABef + \alpha\sqrt{\alpha^2 + A^2e^2 - B^2f^2}}{ABef - \alpha\sqrt{\alpha^2 + A^2e^2 - B^2f^2}}\right) - \frac{(2\alpha^2 + B^2f^2)\sqrt{B^2f^2 - \alpha^2}.\pi}{12ABC}$.

Ex. 3. Corpora *Hyperbolico-hyperbolica* hac continentur æquatione: $A^2x^2 - B^2y^2 - C^2z^2 = \alpha^2$. Formula ergo $zdxdy$ abit in $\frac{dxdy\sqrt{A^2x^2 - \alpha^2 - B^2y^2}}{C}$, qua more solito integrata existit $\frac{dx}{C}\int dy\sqrt{A^2x^2 - \alpha^2 - B^2y^2}$
$= \frac{(A^2x^2 - \alpha^2)dx}{2BC} ArcSin \frac{By}{\sqrt{A^2x^2 - \alpha^2}} + \frac{ydx\sqrt{A^2x^2 - \alpha^2 - B^2y^2}}{2C}$.

Fiat

Fiat $y = \frac{\sqrt{A^2 x^2 - \alpha^2}}{B}$, quo inſerto valore, iterum integrando obtinebitur $\int \frac{(A^2 x^2 - \alpha^2)\pi dx}{4BC} = \frac{(A^2 x^2 - 3\alpha^2)\pi x}{12BC} + C$. Hinc quadrans prodibit ſolidi quæſiti (ſi primo integrale ita corrigatur ut evaneſcat conſtituto $x = \frac{\alpha}{A}$, tum vero conſtituatur $x = e$) $=$ $\frac{(A^2 e^2 - 3\alpha^2)e\pi}{12ABC} + \frac{\alpha^3 \pi}{6ABC}$.

Evolvendo autem formulas . . . $\int \frac{(A^2 x^2 - \alpha^2)dx}{2BC}$ $\text{Arc Sin}\ \frac{Bf}{\sqrt{A^2 x^2 - \alpha^2}}$ $+$ $\int \frac{fdx\sqrt{A^2 x^2 - \alpha^2 - B^2 f^2}}{2C}$ obtinuimus portionem rectangulo $CDEF$ inſiſtentem $=$ . . . .

$$\frac{(A^2 x^2 - 3\alpha^2)x}{6BC}\text{ArcSin}\frac{Bf}{\sqrt{A^2 x^2 - \alpha^2}} + \frac{fx\sqrt{A^2 x^2 - \alpha^2 - B^2 f^2}}{3C}$$

$$- \frac{(3\alpha^2 + B^2 f^2)f}{6AC}\text{Log}\,(Ax + \sqrt{A^2 x^2 - \alpha^2 - B^2 f^2}) +$$

$\frac{\alpha^3}{3ABC}\text{ArcSin}\frac{ABfx}{\sqrt{(\alpha^2 + B^2 f^2)(A^2 x^2 - \alpha^2)}} + C$, quod integrale, ſi terminus E curvam oſculabitur, conſtituto $x = \frac{\sqrt{\alpha^2 + B^2 f^2}}{A}$, nihilo æquari debet, quo caſu erit $C =$

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$$\frac{(2\alpha^2 - B^2f^2)\sqrt{\alpha^2 + B^2f^2}.\pi}{12ABC} + \frac{(3\alpha^2 + B^2f^2)f}{12AC} Log(\alpha^2 + B^2f^2)$$

$- \frac{\alpha^3\pi}{6ABC}$. Facto $x = e$ erit soliditas quæsita $=$

$$\frac{(A^2e^2 - 3\alpha^2)e}{6BC} Arc\,Sin\, \frac{Bf}{\sqrt{A^2e^2 - \alpha^2}} + \frac{ef\sqrt{A^2e^2 - \alpha^2 - B^2f^2}}{3C}$$

$$+ \frac{(3\alpha^2 + B^2f^2)f}{6AC} Log\left(\frac{\sqrt{\alpha^2 + B^2f^2}}{Ae + \sqrt{A^2e^2 - \alpha^2 - B^2f^2}}\right)$$

$$+ \frac{\alpha^3}{3ABC} Arc\,Sin\, \frac{ABef}{\sqrt{(\alpha^2 + B^2f^2)(A^2e^2 - \alpha^2)}} +$$

$$\frac{(2\alpha^2 - B^2f^2)\sqrt{\alpha^2 + B^2f^2}.\pi}{12ABC} - \frac{\alpha^3\pi}{6ABC}.$$

Ex. 4. Solida *Elliptico-parabolicæ* ista continet æquatio: $B^2y^2 + C^2z^2 = ax$, unde $\frac{dx}{C}\int dy\sqrt{ax - B^2y^2} = \frac{ax\,dx}{2BC} Arc\,Sin\, \frac{By}{\sqrt{ax}} + \frac{y\,dx\sqrt{ax - B^2y^2}}{2C}$. Quo integra soliditas hujus obtineatur corporis, fiet $y = \frac{\sqrt{ax}}{B}$, quo inserto valore, integratio repetita præbebit $\int \frac{ax\,dx\,\pi}{4BC} = \frac{ax^2\pi}{8BC} = \frac{ae^2\pi}{8BC} =$ quadranti solidi Elliptico-parabolici.

Portio autem rectangulo *CDEF* insistens e formu-

lis

lis $\int \frac{\alpha x dx}{2BC} ArcSin \frac{Bf}{\sqrt{\alpha x}} + \int \frac{f dx \sqrt{\alpha x - B^2 f^2}}{2C}$ erit petenda, quæ evolutæ dabunt: $\frac{\alpha x^2}{4BC} Arc Sin \frac{Bf}{\sqrt{\alpha x}} + \ldots$ $\frac{(5\alpha x - 2B^2 f^2) f \sqrt{\alpha x - B^2 f^2}}{12 \alpha C} - \frac{B^3 f^4 \pi}{8\alpha C}$ integrali ita correcto ut solidum ad parabolam extensum posito $x = \frac{B^2 f^2}{\alpha}$ evanescat. Facto demum $x = e$ erit solidum quæsitum $= \frac{\alpha e^2}{4BC} ArcSin \frac{Bf}{\sqrt{\alpha e}} + \frac{(5\alpha e - 2B^2 f^2) f \sqrt{\alpha e - B^2 f^2}}{12\alpha C}$ $- \frac{B^3 f^4 \pi}{8\alpha C}$.

Si in æquatione proposita α evanescat, hæc pro Cylindris tam rectis quam scalenis oritur æquatio: $B^2 y^2 + C^2 z^2 = B^2 C^2$. Hinc $dx \int z dy = \frac{2dx}{C} \int B dy \sqrt{C^2 - y^2}$ $= \frac{BCdx}{2} Arc Sin \frac{y}{C} + \frac{Bydx \sqrt{C^2 - y^2}}{2C}$. Fiat $y = C$, quo pacto invenitur quadrans Cylindri $= \int \frac{BC\pi dx}{4} =$ $\frac{1}{4} BC\pi x = \frac{1}{4} BCe\pi$. Portionem autem rectangulo respondentem præbet integrale $\int \frac{Bf dx \sqrt{C^2 - f^2}}{2C} +$

$\int \frac{1}{2} BCdx$

$\int \frac{BCdx}{2} \, Arc\,Sin \frac{f}{C} = \frac{Bfx\sqrt{C^2 - f^2}}{2C} + \frac{BCx}{2} \, Arc\,Sin \frac{f}{C} =$
$\frac{Bef\sqrt{C^2 - f^2}}{2C} + \frac{BCe}{2} \, Arc\,Sin \frac{f}{C}.$

Ex. 5. Solida *Parabolico-hyperbolica* hæc complectitur æquatio $B^2 y^2 - C^2 z^2 = \alpha x$, unde formula $z dx dy = \frac{dx dy \sqrt{B^2 y^2 - \alpha x}}{C}$ ita integrata, ut sola x variabilis spectetur, dabit $\frac{dy}{C} \int dz \sqrt{B^2 y^2 - \alpha x} = -$
$\frac{2dy(B^2 y^2 - \alpha x)^{\frac{3}{2}}}{3\alpha C} + \frac{2B^3 y^3 dy}{3\alpha C}$, integrali ita correcto, ut posito $x = 0$, ipsum quoque in nihilum redigatur. Hinc, facto $\alpha x = B^2 y^2$, prodibit dimidium solidi parabolico-hyperbolici $= \int \frac{2B^3 y^3 dy}{3\alpha C} = \frac{B^3 y^4}{6\alpha C} = \frac{B^3 f^4}{6\alpha C}$. Solidum autem rectangulo insistens erit $= \int \frac{2B^3 y^3 dy}{3\alpha C} - \int \frac{2dy(B^2 y^2 - \alpha e)^{\frac{3}{2}}}{3\alpha C}$
$= \frac{B^3 y^4}{6\alpha C} + \frac{(5\alpha e - 2B^2 y^2) y \sqrt{B^2 y^2 - \alpha e}}{12\alpha C} - \frac{\alpha e^2}{4BC} Log\,(By +$
$\sqrt{(B^2 y^2 - \alpha e)}) - \frac{\alpha e^2}{6\alpha C} + \frac{\alpha e^2}{4BC} Log\,\sqrt{(\alpha e)}$, si quidem rectanguli terminus curvam attingat. Hinc soliditas quæsita, facto $y = f$, prodibit $= \frac{(5\alpha e - 2B^2 f^2) f \sqrt{(B^2 f^2 - \alpha e)}}{12\alpha C}$

$$+ \frac{\alpha e^2}{4BC} Log \left(\frac{\sqrt{(\alpha e)}}{Bf + \sqrt{(B^2 f^2 - \alpha e)}} \right) + \frac{B^2 f^3}{6\alpha C} - \frac{\alpha e^2}{6BC}.$$

Quod si α evanescat in æquatione proposita, naturam *Cylindri hyperbolici* hæc exprimet æquatio: $B^2 y^2 - C^2 z^2 = B^2 C^2$, unde cum sit $\frac{Bdy}{C}\int dx \sqrt{y^2 - C^2}$ $= \frac{Bxdy\sqrt{(y^2 - C^2)}}{C}$, per alteram integrationem, constituto $x = e$, obtinebitur $\int \frac{Bedy\sqrt{(y^2 - C^2)}}{C} = \frac{Bey\sqrt{(y^2 - C^2)}}{2C} -$ $\frac{1}{2}BCe\, Log\, (y + \sqrt{(y^2 - C^2)})$, quod integrale, si ita corrigatur ut posito $y = C$ evanescat, dein vero valor f ipsi tribuatur, dimidium dabit *Cylindri hyperbolici* $=$ $\frac{Bef\sqrt{(f^2 - C^2)}}{2C} + \frac{1}{2}BCe\, Log \frac{C}{f + \sqrt{(f^2 - C^2)}}$. Portionis autem rectangulum tegentis soliditas invenitur, si integrale ita corrigatur ut facto $y = g$ evanescat, quæ ergo erit $= \frac{Be(f\sqrt{(f^2 - C^2)} - g\sqrt{(g^2 - C^2)})}{2C} + \frac{1}{2}BCe \times$ $Log \left(\frac{g + \sqrt{(g^2 - C^2)}}{f + \sqrt{(f^2 - C^2)}} \right)$.

Ex. 6. Æquatio $C^2 z^2 = \alpha y$ *Cylindri parabolici* exhibet naturam; hinc $dy\int zdx = \frac{dy}{C}\int dx \sqrt{(\alpha y)} =$ $\frac{xdy\sqrt{(\alpha y)}}{C}$, constituendo $x = e$, & denuo integrando, erit $\int \frac{edy\sqrt{(\alpha y)}}{C} = \frac{2ey\sqrt{(\alpha y)}}{3C}$, unde dimidium hujus

solifolidi obtinebitur (facto $y = f$) $= \frac{2ef\surd(\alpha f)}{3C}$; fi vero integrale inventum ita corrigatur, ut pofito $y = g$ evanefcat, prodibit foliditas portionis rectangulo infiftentis $= \frac{2e(f\surd(\alpha f) - g\surd(\alpha g))}{3C}$.

Solida *altiorum ordinum* æquationibus definita, pari inveftigare methodo, limites harum pagellarum non permittunt. Quo vero perfpiciatur ufus *coordinatarum obliquangularum*, fequens addamus

§. IV.

PROBLEMA. *Planis, per latera dati Trianguli Sphærici ad centrum ductis, portionem Sphæræ contentam invenire.*

Sint (*Fig.* 2.) ABC, ACD, BCD, plana terna angulum folidum in Centro Sphæræ C formantia, jungantur AB BD, AD. E centro C erigatur normalis CE ad planum per puncta A, B, D transiens & fegmentum Sphæræ $AODNBLA$ abfcindens. Bifecentur Chordæ AB, BD, AD rectis e centro ductis & ad fuperficiem Sphæræ continuatis CFK, CGM, CHP, parique modo e centro E, circuli fectionis $AODNBL$, ducantur EFL, EGN, EHO, & a puncto B demittatur normalis BQ ad AD.

His pofitis, fint, $Sin \frac{1}{2} ACB = AF = \alpha$, $Sin \frac{1}{2} BCD = BG = \beta$ & $Sin \frac{1}{2} ACD = AH = \gamma$, $AC = CB = CD$

 $= a$

$= a =$ *radio Sphæræ;* unde erunt $CF = \sqrt{(a^2 - \alpha^2)}$, $CG = \sqrt{(a^2 - \beta^2)}$, $CH = \sqrt{(a^2 - \gamma^2)}$, $BQ = \frac{\sqrt{(2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\beta^2\gamma^2 - \alpha^4 - \beta^4 - \gamma^4)}}{\gamma}$. Eſt autem $BD : BQ :: 1 : Sin\ ADB\ (= Sin\ AEF) = \frac{\sqrt{(2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\beta^2\gamma^2 - \alpha^4 - \beta^4 - \gamma^4)}}{2\beta\gamma}$, hincque in triangulo AFE, $Tang\ AEF : 1 :: AF : FE = \frac{(\beta^2 - \alpha^2 + \gamma^2)\alpha}{\sqrt{(2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\beta^2\gamma^2 - \alpha^4 - \beta^4 - \gamma^4)}}$. Quare erit *normalis* $CE = \sqrt{\overline{CF^2 - FE^2}}\ (= b) = \sqrt{\left(a^2 - \frac{4\alpha^2\beta^2\gamma^2}{2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\beta^2\gamma^2 - \alpha^4 - \beta^4 - \gamma^4}\right)}$. Porro in triangulo CFE erit $CF : CE :: 1 : Sin\ CFE\ (= Sin\ KFL) = \frac{b}{\sqrt{(a^2 - \alpha^2)}}$ & $CG : CE :: 1 : Sin\ CGE\ (= Sin\ MGN) = \frac{b}{\sqrt{(a^2 - \beta^2)}}$, nec non $CH : CE :: 1 : Sin\ CHE\ (= Sin\ OHP) = \frac{b}{\sqrt{(a^2 - \gamma^2)}}$. Innoteſcit ergo *area* trianguli *plani* $ABD = \sqrt{(2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\beta^2\gamma^2 - \alpha^4 - \beta^4 - \gamma^4)} = \frac{2\alpha\beta\gamma}{\sqrt{(a^2 - b^2)}}$, atque hinc *Pyramidis ABCD* ſoliditas $= \frac{2\alpha\beta\gamma\, b}{3\sqrt{(a^2 - b^2)}}$. Quod ſi *huic* addatur portio *Sphæræ ALBNDO* & a ſumma auferantur portiones *AFBKL*; *BGDMN*

BGDMN & *AHDOP*, residuum exhibebit solidum quæsitum. Portionis autem *ALBNDO* soliditas ea lege, quam hactenus exposuimus, invenitur $=$ $\int \frac{(a^2 - A^2x^2)\pi dx}{4BC} = \frac{(3\alpha^2 - A^2x^2)\pi x}{12BC} + C.$ (vid. *Ex. 1.*) Constans vero quantitas reperitur (ponendo $x = b$, quo casu evanescat soliditas,) $= - \frac{(3\alpha^2 - A^2b^2)b\pi}{12BC}$. Quare tota soliditas portionis Sphæricæ *ALBNDOA* erit $= (2a^3 - 3a^2b + b^3)\frac{\pi}{3}$, facto $\frac{\alpha}{A} = \frac{\alpha}{B} = \frac{\alpha}{C} = x = a$.

Quo demum pateat portio *AFBKL* sive ejus dimidium *AFKL*, notetur esse angulos *AFK*, *AFL* *rectos*, angulum vero *KFL obliquum*, qui ponatur (*Fig. 1.*) $= APG = m$. Hoc ergo casu formula *generalis* abit in $zdxdy \, Sin \, m$. Sint, $CP = x$, $PM = y$, $QM = z$ & *radius Sphæræ* $AC = a$. Demittatur a puncto *M* normalis *MV*, quo facto erit $MV = y \, Sin \, m$ & $PV = y \, Cos \, m$, hincque $CV = x + y \, Cos \, m$. Æquatis vero *quadratis* coordinatarum *orthogonalium* *CV*, *MV*, *MQ* cum *quadrato radii*, hæc pro Sphæra emergit æquatio: $x^2 + 2xy \, Cos \, m + y^2 + z^2 = a^2$. Hinc erit $zdxdy \, Sin \, m = dxdy \, Sin \, m \sqrt{(a^2 - x^2 - 2xy \, Cos \, m - y^2)}$, quæ formula pari ac supra methodo expediri potest. Quo primum integrale inveniatur sumta sola y variabili, fiat substituendo $v = \frac{x \, Cos \, m + y}{\sqrt{(a^2 - x^2 \, Sin \, m^2)}}$, unde inveniuntur $dy = dv\sqrt{(a^2 - x^2 \, Sin\, m^2)}$ & . . $\sqrt{(a^2 - x^2 - 2xy\, Cos\, m - y^2)} = \sqrt{(a^2 - x^2 \, Sin\, m^2)} \times \sqrt{(1 - v^2)}$, qui valores dabunt $dx \, Sin\, m \times$ . . $\int dy \sqrt{(a^2 - x^2 - 2xy\, Cos\, m - y^2)} = (a^2 - x^2 \, Sin\, m^2)\, dx \, Sin\, m \times \int dv \sqrt{(1 - v^2)} = \frac{1}{2}(x\, Cos\, m + y)\, Sin\, m.\, dx \times$ . . $\sqrt{(a^2 - x^2 - 2xy\, Cos\, m - y^2)} + \frac{1}{2}(a^2 - x^2 \, Sin\, m^2)\, dx\, Sin\, m \times Arc\, Sin \frac{x\, Cos\, m + y}{\sqrt{(a^2 - x^2 \, Sin\, m^2)}} - \frac{1}{2} x dx \, Sin\, m\, Cos\, m. \sqrt{(a^2 - x^2)} - \frac{(a^2 - x^2 \, Sin\, m^2)\, dx\, Sin\, m}{2} Arc Sin \frac{x\, Cos\, m}{\sqrt{(a^2 - x^2 \, Sin\, m^2)}}$, integrali ex natura quæstionis ita correcto, ut posito $y = 0$, ipsum quoque evanescat. Porrigatur y usque ad peripheriam AEG, quo pacto erit $y = PG = - x\, Cos\, m + \sqrt{(a^2 - x^2 \, Sin\, m^2)}$, qua functione pro y inserta, erit $\int \frac{(a^2 - x^2 \, Sin\, m^2)\, Sin\, m.\, \pi dx}{4} -$ . . . . $\int \frac{1}{2}(a^2 - x^2 \, Sin\, m^2)\, dx\, Sin\, m.\, Arc\, Sin \frac{x\, Cos\, m}{\sqrt{(a^2 - x^2 \, Sin\, m^2)}} - \int \frac{1}{2} x dx\, Sin\, m\, Cos\, m. \sqrt{(a^2 - x^2)} = \frac{1}{6}(3a^2 - x^2 \, Sin\, m^2) \times x\, Sin\, m \left(\frac{\pi}{2} - Arc\, Sin \frac{x\, Cos\, m}{\sqrt{(a^2 - x^2 \, Sin\, m^2)}}\right) - \frac{1}{6} x^2 \, Sin\, m \times Cos\, m. \sqrt{(a^2 - x^2)} - \frac{1}{3} a^3 \, Arc\, Sin \frac{\sqrt{(a^2 - x^2)}\, Sin\, m}{\sqrt{(a^2 - x^2 \, Sin\, m^2)}} + C.$ Quare facto $Sin\, m = \frac{b}{\sqrt{(a^2 - \alpha^2)}}$, $Cos\, m =$

$\sqrt{(a^2}$

$\frac{\sqrt{(a^2-\alpha^2-b^2)}}{\sqrt{(a^2-\alpha^2)}}$ & extenſo x usque ad A, integrale: (ita correctum, ut poſito $x = \sqrt{(a^2 - \alpha^2)} = CF$, (*Fig.* 2.) evaneſcat) $\frac{1}{6}\alpha b\sqrt{(a^2 - \alpha^2 - b^2)} -$ . . . $\frac{1}{6}(3a^2 - b^2)b\left(\frac{\pi}{2} - Arc\,Sin\,\frac{\sqrt{(a^2 - \alpha^2 - b^2)}}{\sqrt{(a^2 - b^2)}}\right) +$ $\frac{1}{3}a^3\,Arc\,Sin\,\frac{\alpha b}{\sqrt{((a^2-b^2)(a^2-\alpha^2))}}$ dimidium præbet portionis $AFBKL$, unde tota $AFBKL = \frac{1}{3}\alpha b\sqrt{(a^2-\alpha^2-b^2)}$ $- \frac{1}{3}(3a^2 - b^2)\,b.\,Arc\,Sin\,\frac{\alpha}{\sqrt{(a^2 - b^2)}} +$ . . $\frac{2}{3}a^3\,Arc\,Sin\,\frac{\alpha b}{\sqrt{((a^2-b^2)(a^2-\alpha^2))}}$, facta debita reductione. Pari omnino modo invenitur $BGDMN =$ $\frac{2}{3}\beta b\sqrt{(a^2-\beta^2-b^2)} - \frac{1}{3}(3a^2-b^2)b.\,Arc\,Sin\,\frac{\beta}{\sqrt{(a^2-b^2)}}$ $+\frac{2a^3}{3}\,Arc\,Sin\,\frac{\beta b}{\sqrt{((a^2-b^2)(a^2-\beta^2))}}$, & $AHDOP =$ $\frac{1}{3}\gamma b\sqrt{(a^2-\gamma^2-b^2)} - \frac{1}{3}(3a^2-b^2)\,b.\,Arc\,Sin\,\frac{\gamma}{\sqrt{(a^2-b^2)}}$ $+\frac{2a^3}{3}\,Arc\,Sin\,\frac{\gamma b}{\sqrt{((a^2-b^2)(a^2-\gamma^2))}}$. Hinc ergo erit *ſolidum quæſitum* $CAKBMDP = \frac{2\alpha\beta\gamma.b}{3\sqrt{(a^2-b^2)}} +$ $(2a^3-3a^2b+b^3)\frac{\pi}{3} - \frac{\alpha b\sqrt{(a^2-\alpha^2-b^2)}}{3} - \frac{\beta b\sqrt{(a^2-\beta^2-b^2)}}{3}$

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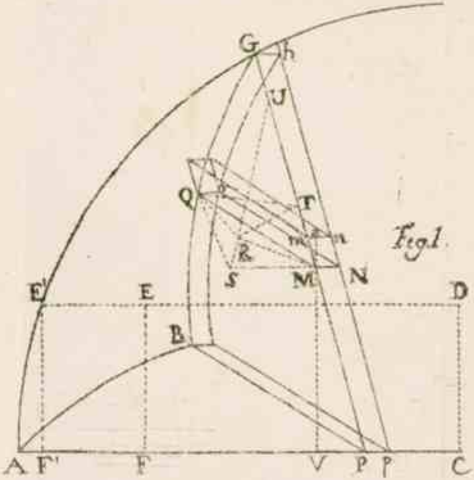

Fig. 1.

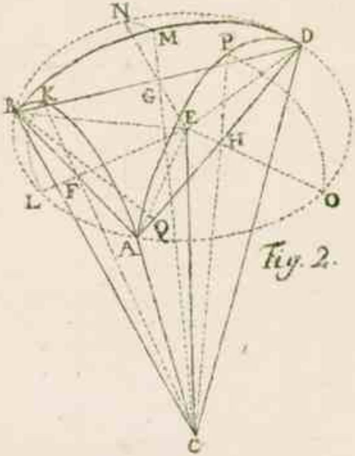

Fig. 2.

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$$- \frac{\gamma b \sqrt{(a^2 - \gamma^2 - b^2)}}{3} + \frac{(3a^2 - b^2)b}{3} \left(Arc\,Sin\,\frac{\alpha}{\sqrt{(a^2 - b^2)}} \right.$$

$$\left. + Arc\,Sin\,\frac{\beta}{\sqrt{(a^2 - b^2)}} + Arc\,Sin\,\frac{\gamma}{\sqrt{(a^2 - b^2)}} \right) +$$

$$\frac{2a^3}{3} \left(ArcSin \frac{\alpha b}{\sqrt{((a^2 - b^2)(a^2 - \alpha^2))}} + ArcSin \frac{\beta b}{\sqrt{((a^2 - b^2)(a^2 - \beta^2))}} \right.$$

$\left. + ArcSin \frac{\gamma b}{\sqrt{((a^2 - b^2)(a^2 - \gamma^2))}} \right)$, quæ expresſio reducitur ad hanc: $\frac{2a^3}{3} \left(\pi - Arc\,Sin \frac{\alpha b}{\sqrt{((a^2 - b^2)(a^2 - \alpha^2))}} - \right.$

$$\left. Arc\,Sin \frac{\beta b}{\sqrt{((a^2 - b^2)(a^2 - \beta^2))}} - ArcSin \frac{\gamma b}{\sqrt{((a^2 - b^2)(a^2 - \gamma^2))}} \right).$$